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Numerical Solution of Optimal Control Problems with Discrete-Valued System Parameters ⋆

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Abstract. In this paper, we propose a new approach to solve a class of optimal control problems involving discrete-valued system parameters. The basic idea is to formulate a problem of this type as a combination of a discrete global optimization problem and a standard optimal control problem, and then solve it using a two-level approach. Numerical results show that the proposed method is efficient and capable of finding optimal or near optimal solutions.

1. Introduction

Optimal Control Problems arise in many disciplines such as engineering, economics, physics and the biomedical sciences. Many optimal control problems involve system (decision) parameters which are restricted to a set of discrete values [11, 13]. These discrete sets typically arise because a design is only allowed to use standard-sized components or parts which are readily available, such as standard size drill collars used in the oil and gas industry or the finite number of gear ratios available in transmission. The essential nature of these problems is that a cost functional $g_0(\boldsymbol{u}, \boldsymbol{z})$, depending on a control function $\boldsymbol{u}(t) \in \mathbb{R}^m$ and a system parameter vector $\boldsymbol{z} \in \mathbb{R}^p$, is to be minimized subject to a dynamical system and a set of constraints with the restriction that $\boldsymbol{z} \in \boldsymbol{Z}$, where $\boldsymbol{Z}$ is a discrete subset of $\mathbb{R}^p$. Note that the discrete nature of the set $\boldsymbol{Z}$ necessarily means that we are considering a class of nonconvex optimal control problems here.

In [13], a closely related class of optimal control problems involving differential algebraic systems and integer variables is considered. The proposed solution strategy is to fully discretize the problem by an implicit Runge-Kutta scheme, although no numerical results are given.

In this paper, we propose a different technique. Following an approach similar to that in [17], where an ordinary optimal control problem is solved by optimizing over two levels, we decompose the problem into a two-level optimization problem. At the 'upper' level, we are faced with a purely discrete optimization problem, which may be solved using any suitable discrete optimization approach (such as

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Simulated Annealing, Branch and Bound, or Genetic Algorithms [2, 4, 5, 16, 22]). Amongst these, stochastic methods generally lead to better results than deterministic ones [18]. For this reason, we develop a simulated annealing approach with memory to solve the upper level problem. For each iteration of the upper level optimization, we then solve a lower level problem where the system parameters are fixed. This lower level problem is simply a standard optimal control problem which may be solved by any of a variety of suitable techniques [7, 8, 12, 14, 15, 19, 20, 23, 25]. In this work, we follow the control parameterization approach, because of its robustness and readily available software.

Note that the class of problems under consideration is distinct from the class of *discrete valued optimal control problems*, where it is the control functions which are assumed to only take on values from a discrete set. For application examples of this type of problem, see [6, 21]. An effective numerical solution method for these types of problems, based on the *Control Parameterization Enhancing Transform (CPET)*, is proposed in [10].

The rest of the paper is organized as follows. In the next section, we formulate the class of problems under consideration. Simulated annealing with memory for the upper level optimization is discussed in Section 3. An efficient cooling schedule for the simulated annealing approach is developed in Section 4. In Section 5, an example is solved to demonstrate the efficiency of the proposed method.

2. Problem Statement

Consider a process described by the following system of differential equations defined on $[0, t_f]$:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{z}) \tag{1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{2}$$

where t_f is a fixed terminal time, $\boldsymbol{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $\boldsymbol{u} = [u_1, \ldots, u_m]^T \in \mathbb{R}^m$ and $\boldsymbol{z} = [z_1, \ldots, z_p]^T \in \mathbb{R}^p$ are, respectively, the state, control, and system parameter vectors, while $\boldsymbol{f} = [f_1, \ldots, f_n]^T \in \mathbb{R}^n$ is a continuously differentiable function with respect to all its arguments and $\boldsymbol{x}_0$ is a given vector. Define

$$\boldsymbol{U} = \{\boldsymbol{u} = [u_1, \ldots, u_m]^T \in \mathbb{R}^m : c_i \le u_i \le d_i,\ i = 1, \ldots, m\}$$

$$\boldsymbol{Z} = \{\boldsymbol{z} = [z_1, \ldots, z_p]^T \in \mathbb{R}^p :\ z_i \in D_i, \quad i = 1, \ldots, p\}$$

where $D_i = \{\delta_1^i, \delta_2^i, \ldots, \delta_{M_i}^i\}$, $i = 1, \ldots, p$, are given discrete sets containing M_i real entries, respectively.

Any Borel measurable function $\boldsymbol{u} : [0, t_f] \longrightarrow \boldsymbol{U}$ is called an admissible control. Let $\mathcal{U}$ be the class of all admissible controls. For each $(\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \times \boldsymbol{Z}$, let $\boldsymbol{x}(\cdot|\boldsymbol{u}, \boldsymbol{z})$ denote the corresponding solution of the system (1)–(2). Our optimal control problem may now be formally stated as:

Given the dynamical system (1)–(2), find a $(\boldsymbol{u}, z) \in \mathcal{U} \times \boldsymbol{Z}$ such that the cost functional

$$g_0(\boldsymbol{u}, z) = \Phi_0(\boldsymbol{x}(t_f|\boldsymbol{u}, z), z) + \int_0^{t_f} \mathcal{L}_0(t, \boldsymbol{x}(t|\boldsymbol{u}, z), \boldsymbol{u}(t), z)dt, \tag{3}$$

is minimized subject to the equality constraints:

$$g_i(\boldsymbol{u}, z) = \Phi_i(\boldsymbol{x}(t_f|\boldsymbol{u}, z), z) + \int_0^{t_f} \mathcal{L}_i(t, \boldsymbol{x}(t|\boldsymbol{u}, z), \boldsymbol{u}(t), z)dt = 0,$$
$$i = 1, \ldots, N_e, \tag{4}$$

and the inequality constraints:

$$g_i(\boldsymbol{u}, z) = \Phi_i(\boldsymbol{x}(t_f|\boldsymbol{u}, z), z) + \int_0^{t_f} \mathcal{L}_i(t, \boldsymbol{x}(t|\boldsymbol{u}, z), \boldsymbol{u}(t), z)dt \geq 0,$$
$$i = N_e, \ldots, N, \tag{5}$$

where Φ_i, $i = 0, 1, \ldots, N$, and $\mathcal{L}_i, i = 0, 1, \ldots, N$, are real valued functions which are continuously differentiable with respect to all their arguments. Let this optimal control problem be referred to as Problem (P).

The two level decomposition of Problem (P) can be described as follows. Define Problem (P_1) as

$$\min_{z \in z} \bar{g}_0(z)$$

where

$$\bar{g}_0(z) = \min_{\boldsymbol{u} \in \mathcal{U}} g_0(\boldsymbol{u}, z),$$

subject to the constraints (4) and (5). The latter problem, which involves a fixed z, is referred to as Problem (P_2). Problem (P_2) is a standard optimal control problem.

Clearly, each iteration in the solution strategy for Problem (P_1) requires an optimal solution of Problem (P_2). Here, we use the software MISER 3.2 [7, 8], which is based on the control parameterization approach, to solve Problem (P_2). Note that there are other efficient optimal control softwares that may also be used at this point. Problem (P_1) can then be solved using a global optimization technique such as simulated annealing.

3. Simulated Annealing with Memory

The Simulated annealing (SA) process is one of the more effective techniques in the area of global optimization and a detailed analysis can be found in the references already cited. Essentially, SA is a search procedure where each new point found is either accepted or rejected according to a probability (governed by the so-called

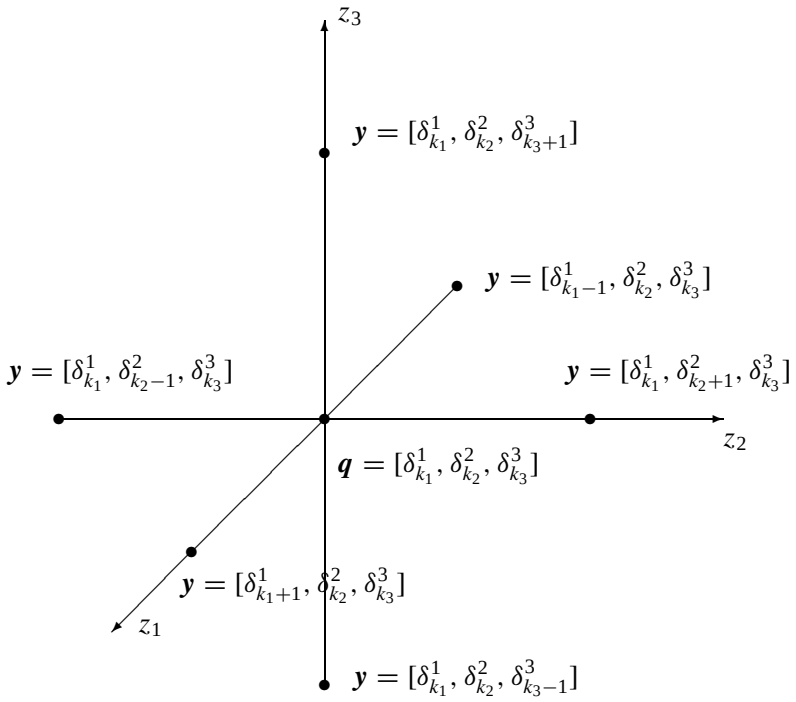


Figure 1. The neighborhood of $\boldsymbol{q}$ in a 3-dimensional problem

cooling schedule T) which changes at the various stages. At any one stage, the value of T controls the acceptance rate of new points in the search which result in an increase of the objective function value, *i.e.* the probability of taking an 'uphill' step. During the first stage, the value of the cooling schedule, T_0, is large. This leads to a high probability of taking an uphill step and the acceptance rate is close to 1. In subsequent stages, T is decreased, meaning that there is a lower probability of taking an uphill step. This results in a lower acceptance rate. When T is close to 0, the probability of taking an uphill step is extremely low. Therefore, the *cooling schedule, T,* is also known as the *control parameter*. This basic feature of SA allows it to effectively avoid local minima, whilst still exhibiting the favorable features of a local search.

To discuss SA in more detail, we first define a neighborhood structure. Let $\boldsymbol{q} \in \boldsymbol{Z}$. Suppose that $\boldsymbol{q} = [\delta^1_{k_1}, \delta^2_{k_2}, \ldots, \delta^p_{k_p}]^T$, then the neighborhood of $\boldsymbol{q}$ is defined by:

$$\mathbb{N}_{\boldsymbol{q}} = \{\boldsymbol{y} = [y_1, \ldots, y_p]^T \in \boldsymbol{Z} : \ y_i = \delta^i_{k_i+1} \text{ or } y_i = \delta^i_{k_i-1} \text{ for some } i = 1, \ldots, p\}.$$

For illustration, a 3-dimensional neighborhood is depicted in Figure 1. Clearly, the size of the neighborhood, Θ, is equal to $2p$ unless $\boldsymbol{q}$ is on the boundary of $\boldsymbol{Z}$. The Metropolis acceptance criterion (see [1]) determines whether $\boldsymbol{y}$ is accepted from $\boldsymbol{q}$

at the control level T by applying the following acceptance probability

$$A_{\boldsymbol{q}\,\boldsymbol{y}}(T) = \begin{cases} \exp\left\{-\frac{(g_0(\boldsymbol{y})-g_0(\boldsymbol{q}))}{T}\right\}, & \text{if } g_0(\boldsymbol{y}) > g_0(\boldsymbol{q}), \\ 1, & \text{otherwise.} \end{cases}$$

Here, the next possible iterate $\boldsymbol{y}$ is generated from the current iterate $\boldsymbol{q}$ according to the generating probability:

$$G_{\boldsymbol{q}\,\boldsymbol{y}} = \frac{1}{\Theta}\mathcal{X}(\mathbb{N}_{\boldsymbol{q}})(\boldsymbol{y}),$$

where $\mathcal{X}$ is a uniform distribution on $\mathbb{N}_{\boldsymbol{q}}$. In other words, the next possible iterate of Problem (P_1) is chosen randomly from the neighborhood $\mathbb{N}_{\boldsymbol{q}}$. By comparing the value of $A_{\boldsymbol{q}\,\boldsymbol{y}}$ with a random number generated from a uniform distribution on the interval [0, 1], we then decide whether $\boldsymbol{y}$ is accepted as the next iterate or not.

The SA process sometimes accepts iterates which are worse than a current iterate. During a particular SA run, it is therefore possible for the final iterate to be worse than another iterate found at some point during the run. In fact, the SA algorithm is a randomization device which, by means of an acceptance-rejection criterion, allows some ascent steps during the optimization process. It is quite possible that during a particular stage, the procedure will visit the optimal solution, but due to the acceptance-rejection mechanism, it will move on from this solution and finish at a suboptimal solution. The algorithm developed in this paper takes such behavior into account and adds a memory $g_0^m(\boldsymbol{u}^m, \boldsymbol{z}^m)$ to the process. At each stage of the process corresponding to a fixed control level T, a Markov chain is created according to the transition probability $A_{\boldsymbol{q}\,\boldsymbol{y}}$. During the execution of each Markov chain, if a point generates a lesser objective function value than the memory, then the memory is updated. At the conclusion of the SA process, the memory $g_0^m(\boldsymbol{z}^m, \boldsymbol{z}^m)$ is taken to be the optimal solution $g_0^*(\boldsymbol{u}^*, \boldsymbol{z}^*)$.

4. Cooling Schedules

The rate of convergence of the algorithm is determined by

- the choice of the initial control parameter T_0,
- the decrement, α_i, of the control parameters from T_i to T_{i+1} and the length of the corresponding Markov chains, L_i,
- the number of transitions generated at the $i-th$ iteration of the metropolis algorithm.

We discuss the choices of these parameters below.

4.1. INITIAL VALUE OF THE CONTROL PARAMETER

To discuss the choice of the initial value of the control parameter we first define the *acceptance ratio,* $\chi(T)$, by

$$\chi(T) = \left.\frac{\text{number of accepted transitions}}{\text{number of proposed transitions}}\right|_T .$$

The initial control parameter T_0 should be large enough to allow nearly all proposed transitions to be accepted in the first Markov chain. This is achieved by requiring that the initial acceptance ratio, $\chi(T_0)$, is close to 1. Let $\{\xi_{z_i,z_{i+1}},\ i = 1,\ldots,r\}$ be a random sequence of r transitions. Let m_1 denote the number of transitions in the sequence with $g_0(z_{i+1}) \leq g_0(z_i)$, and let m_2 be the number of transitions with $g_0(z_{i+1}) > g_0(z_i)$. Let $\eta(\xi_{z_i,z_{i+1}})$ denote the characteristic function of the transition $\xi_{z_i,z_{i+1}}$ defined by:

$$\eta(\xi_{z_i,z_{i+1}}) = \begin{cases} 0, & A(\xi_{z_i,z_{i+1}}) < \beta, \\ 1, & \text{otherwise}, \end{cases} \tag{6}$$

where

$$A(\xi_{z_i,z_{i+1}}) = \begin{cases} \exp\left\{-\frac{g_0(z_{i+1})-g_0(z_i)}{T_0}\right\}, & g_0(z_{i+1}) > g_0(z_i), \\ 1, & \text{otherwise}. \end{cases} \tag{7}$$

Here, β is a random number in the interval [0,1]. Then, we have the following theorem.

THEOREM 4.1 *Suppose* $\{\xi_{z_i,z_{i+1}},\ i = 1,\ldots,r\}$ *is a random sequence of* r *independent transitions, which includes* m_1 *cost-decreasing or cost-equal transitions and* $m_2 = r - m_1$ *cost-increasing transition, and* $\eta(\xi_{z_i,z_{i+1}})$ *is the characteristic function of the transitions* $\xi_{z_i,z_{i+1}}$ *as defined in (6) – (7). Let* $S_r = \sum_{i=1}^{r} \eta(\xi_{z_i,z_{i+1}})$. *Then we have*

$$E(S_r) = E(m_1) + (r - E(m_1)) \cdot E\left\{\exp\left\{-\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\},$$

where $\xi_{z_i^{m_2},z_{i+1}^{m_2}}$ *is any cost-increasing transition in the sequence.*

Proof. We first find the conditional expectation of S_r, given the value of m_1, as

$$\begin{aligned} E(S_r|m_1 = l) &= E(\eta(\xi_{1,2}) + \ldots + \eta(\xi_{n,n+1})|m_1 = l) \\ &= E(\eta(\xi_{1,2})) + \ldots + E(\eta(\xi_{n,n+1})) \\ &\quad (\text{with } m_1 = l, \text{ and } m_2 = r - l) \\ &= l + (r - l)\backslash P\left\{A(\xi_{z_i,z_{i+1}}) \geq \beta |\ g_0(z_{i+1}) > g_0(z_i)\right\}, \end{aligned}$$

where $E(\cdot)$, $P(\cdot)$ denote expectation and probability, respectively.

Let γ denote the random variable $\exp\{-\frac{g_0(z_{i+1})-g_0(z_i)}{T_0}\}$, associated with a cost-increase transition $\xi_{z_i,z_{i+1}}$, and let $f(\gamma)$ denote the probability density function

of γ. Further, let ν denote a random variable with a uniform distribution, defined on the interval [0,1]. Clearly, ν and γ are independent and have joint distribution $f(\gamma)$. Therefore, we have

$$P\left\{A(\xi_{z_i,z_{i+1}}) \geq \beta |\ g_0(z_{i+1}) > g_0(z_i)\right\} = \int_0^1 f(\gamma)d\gamma \int_0^\gamma d\nu$$
$$= E(\gamma) = E\left\{\exp\left\{-\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\},$$

where $\xi_{z_i^{m_2},z_{i+1}^{m_2}}$ is a cost-increasing transition.

$$\text{Therefore,} \quad E(S_r) = \sum_{l=1}^{r} E(S_r|m_1 = l)\backslash P(m_1 = l)$$
$$= E(m_1) + E(r - m_1)\backslash E\left\{\exp\left\{-\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\}$$
$$= E(m_1) + (r - E(m_1))\backslash E\left\{\exp\left\{-\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\}.$$

Clearly, the exponential function is convex. Hence, by Jensen's Inequality ([3]), we have

$$E\left\{\exp\left\{-\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\} \geq \exp\left\{-E\left\{\frac{g_0(z_{i+1}^{m_2}) - g_0(z_i^{m_2})}{T_0}\right\}\right\}$$

According to the above analysis, the initial acceptance ratio can be approximated by the following expression:

$$\chi_0 \approx \frac{m_1 + m_2 \cdot \exp\left(\frac{-\overline{\Delta g}^{(+)}}{T_0}\right)}{m_1 + m_2}$$

where $\overline{\Delta g}^{(+)}$ is the average difference in cost over m_2 cost-increasing transitions. Therefore we choose an initial acceptance ratio χ_0 rather than an initial control parameter T_0. T_0 can then be approximated as:

$$T_0 = \frac{-\overline{\Delta g}^{(+)}}{ln\left(\frac{m_2}{m_2 \cdot \chi_0 - m_1 \cdot (1-\chi_0)}\right)} \tag{8}$$

Hence, the initial control parameter T_0 and initial value of z_0 can be calculated using following procedure:

1. Create a random sequence of transitions $\{\xi_{z_i,z_{i+1}}, i = 1, \ldots, r\}$.
2. Calculate the initial value T_0 of control parameter from (8).
3. Find the smallest value, $g_0(z_i^0)$, in the sequence $\{g_0(z_i), i = 1, \ldots, r+1\}$. The corresponding parameter z_i^0 is used as the initial value of z_0.

4.2. LENGTH OF MARKOV CHAIN AND CORRESPONDING DECREMENT OF THE CONTROL PARAMETER

At stage i of the SA process with control parameter T_i, the length of the Markov chain is based on the requirement that quasi equilibrium is to be restored (see [1, 24]). The number of transitions needed to achieve this is calculated from the intuitive argument that quasi equilibrium will be restored after acceptance of at least some fixed number of transitions. It is clear that large decrement in T_i will require longer Markov chain length in order to restore quasi equilibrium at the next value of the control parameter T_{i+1}. In our approach, we fix the length of Markov chain as follows.

$$L = L_0\ \Theta,$$

where L_0 denotes a constant and Θ is as defined in the previous section. The decrement in T from stage i to stage $i + 1$ is given by

$$T_{i+1} = \alpha_i\ T_i,$$

where α_i is decided according to the acceptance ratio $\chi(T_i)$ of the i_{th} stage. We use the following algorithm.

Algorithm:

1. Calculate the acceptance ratio, $\chi(T_i)$, of the i_{th} stage.
2. Calculate the weighted acceptance ratio, $\chi^w(T_i)$, of the i_{th} stage by

   $$\chi^w(T_i) = \zeta^i\ \chi(T_i),$$

   where $\zeta > 1$ is a constant.
3. Calculate the decrement α_i in the following manner:

if	$\chi^w(T_i)$ less than 0.59,	then α_i = 1.15
elseif	$\chi^w(T_i)$ less than 0.69,	then α_i = 1.1
elseif	$\chi^w(T_i)$ less than 0.79,	then α_i = 1.05
elseif	$\chi^w(T_i)$ less than 0.84,	then α_i = 1.0
elseif	$\chi^w(T_i)$ less than 0.88,	then α_i = 0.95
elseif	$\chi^w(T_i)$ less than 0.92,	then α_i = 0.90
else		α_i = 0.85

Here, we fix the length of the Markov chain at stage i and let the decrement α_i vary to achieve the requirement for restoring quasi equilibrium at stage $i + 1$. When the weighted acceptance ratio $\chi^w(T_i)$ gets smaller, the decrement α_i will clearly become larger to insure restoration of the equilibrium at the next stage. $\chi^w(T_0)$ is also used to prevent underestimation of the initial value of the control parameter T_0. When the iteration number i gets larger, the weight ζ^i will increase. Hence, a small decrement α_i is allowed when the acceptance ratio is small.

4.3. STOPPING CRITERIA

The stopping criterion proposed by Kirkpatrick, Gelatt and Vecchi [9] is used in our approach. That is, the execution of the algorithm is terminated if the value of the cost function of the solution in the last stage is unchanged for s consecutive chains.

Summarizing, we have derived a cooling schedule depending on four parameters, these being the initial acceptance ratio χ_0, the constant L_0 for the fixed length of the Markov chains, the constant ζ for control the decrement of the control parameters and the stopping parameter s.

5. Numerical Experiments

To demonstrate the efficiency of the approach described in the previous sections, a numerical example was solved. All computations were performed in Fortran 77 double precision on a Unix workstation.

Example. We are given the following system of differential equations defined on [0, 1]:

$$\begin{aligned}
\dot{x}_1 &= 0.3x_2u_1 + z_1 \\
\dot{x}_2 &= 0.1x_1x_3z_2 + 0.25x_2 \\
\dot{x}_3 &= 0.15x_4u_2 + z_3 \\
\dot{x}_4 &= 0.2x_3x_1z_4 + 0.15x_4,
\end{aligned}$$

with initial condition $\boldsymbol{x}_0 = (0, 0, 0, 0)^T$, where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$, $\boldsymbol{u} = (u_1, u_2)^T \in \mathbb{R}^2$ and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)^T \in \mathbb{R}^4$ are, respectively, the state, control and system parameters. We define

$$\begin{aligned}
\boldsymbol{U} &= \{\boldsymbol{u} = (u_1, u_2)^T \in \mathbb{R}^2 : \quad 0 \le u_i \le 2, \quad i = 1, 2\}, \\
\boldsymbol{Z} &= \{\boldsymbol{z} = (z_1, z_2, z_3, z_4)^T \in \mathbb{R}^4 : z_i \in \{0, 1, 2, \ldots, 9\}, i = 1, 2, 3, 4\}.
\end{aligned}$$

Further, we define $\mathcal{U}$ as in Section 2. The optimal control problem is:

Given the above dynamic system, find $(\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \times \boldsymbol{Z}$, such that

$$g(\boldsymbol{u}, \boldsymbol{z}) = -\sum_{i=1}^{7}\{(\boldsymbol{x}(t_f) - \boldsymbol{a}_i)^T(\boldsymbol{x}(t_f) - \boldsymbol{a}_i) + c_i\}^{-1} + \int_0^{t_f}(x_3^2 + x_4^2 + u_2^2)dt$$

is minimized, where $t_f = 1$ and the remainder problem parameters are given in Table I. For Problem (P_1), there are 10,000 feasible points. These are depicted in Figure 2, which plots the corresponding objective function values. We can see that there are several local minima. Further, the global minimum is $g_0(\boldsymbol{u}^*, \boldsymbol{z}^*) =$ -3.15274, corresponding to the optimal system parameter $\boldsymbol{z}^* = (3, 6, 4, 4)^T$ and optimal control $\boldsymbol{u}^*$ depicted in Figure 3.

Table I. The parameters for the example.

i	$a_{i,j}$				c_i
1	4	4	4	4	0.1
2	1	1	1	1	0.2
3	8	8	8	8	0.2
4	6	6	6	6	0.4
5	3	7	3	7	0.4
6	2	9	2	9	0.6
7	5	5	3	3	0.3

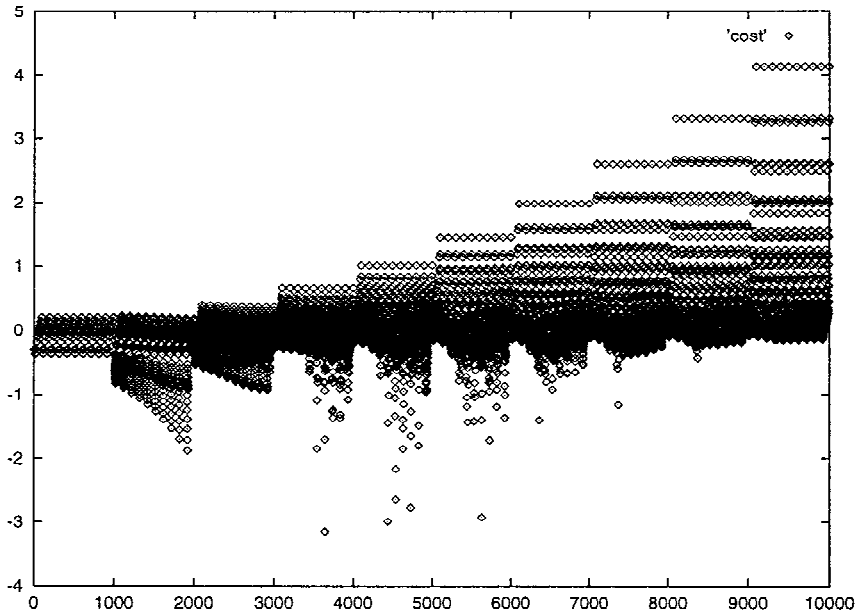
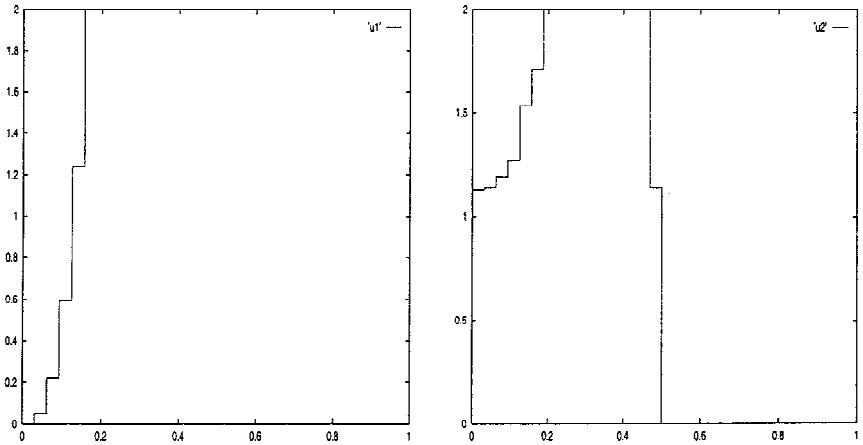


Figure 2. The cost function plotted against $x = 1000z_1 + 100z_2 + 10z_3 + z_4$.

The neighborhood of any possible solution $\boldsymbol{q} \in \boldsymbol{Z}$ for the Problem (P_1) is

$$\mathbb{N}\boldsymbol{q} = \{\boldsymbol{y} = (y_1, y_2, y_3, y_4)^T \in \boldsymbol{Z} : ||\boldsymbol{y} - \boldsymbol{q}|| = 1\}.$$

To determine the cooling schedule, we use the following parameters: $\chi_0 = 0.9$, $L_0 = 2.5$, $\zeta = 1.2$ and $s = 1$. Initially, a random sequence is created to estimate the initial control parameter and yield a starting point. The results obtained for different initial point $\boldsymbol{z}_s$ of the random sequence are listed in the Table II. From the table, we see that the approach gives an optimal or a near optimal solution with about 240 to 520 iterations of solving Problem (P_2).



(a) The control u_1^* (b) The control u_2^*

Figure 3. Computed optimal control $\boldsymbol{u}^*$.

Table II. Results for different start points z_s of random sequences.

z_s	z^*	g_0^*	n_{P_2}
$(1, 1, 1, 1)^T$	$(4, 7, 3, 3)^T$	-2.77260	220
$(2, 2, 2, 1)^T$	$(5, 6, 3, 3)^T$	-2.92316	220
$(3, 3, 3, 1)^T$	$(4, 4, 4, 3)^T$	-2.98909	280
$(4, 4, 4, 1)^T$	$(3, 6, 4, 4)^T$	-3.15274	520
$(5, 5, 5, 1)^T$	$(4, 4, 4, 3)^T$	-2.98909	240
$(6, 6, 6, 6)^T$	$(4, 4, 4, 3)^T$	-2.98909	380
$(7, 7, 7, 7)^T$	$(4, 4, 4, 3)^T$	-2.98909	280
$(8, 8, 8, 8)^T$	$(1, 9, 1, 9)^T$	-1.87676	380
$(9, 9, 9, 9)^T$	$(5, 6, 3, 3)^T$	-2.92316	240

6. Conclusions

In this paper the optimal control problem with discrete-valued system parameter is treated as a global discrete-valued optimization problem where each iteration involves the solution of a standard optimal control problem. Numerical results show that the approach is efficient, displays fast convergence, and gives optimal or near optimal solutions.

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